Phase transitions in a disordered system in and out of equilibrium

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The equilibrium and non–equilibrium disorder induced phase transitions are compared in the random-field Ising model (RFIM). We identify in the demagnetized state (DS) the correct non-equilibrium hysteretic counterpart of the $T = 0$ ground state (GS), and present evidence of universality. Numerical simulations in $d = 3$ indicate that exponents and scaling functions coincide, while the location of the critical point differs, as corroborated by exact results for the Bethe lattice. These results are of relevance for optimization, and for the generic question of universality in the presence of disorder.

PACS numbers:

Similarities and differences between equilibrium and non–equilibrium states in disordered systems have been widely studied both for their conceptual importance and because the presence of randomness often provides prototypical examples of complex optimization problems [1]. There are also many applications in the physics of materials, where this dichotomy is met, together often with concepts such as aging and glassiness. The central issue is to understand whether the equilibrium properties of disordered systems provide a faithful representation of the non–equilibrium states in which the system is likely to be found in practice. In optimization terms, the question is what is the relation between an approximate solution and the optimal one.

A disordered system can be non-trivial even at zero temperature due to the presence of a complex energy landscape. The properties of the ground-state (GS) are often difficult to determine analytically, and numerical evaluation becomes computationally prohibitive for large systems in particular and for some problems like spin glasses, in general. Non-equilibrium dynamics brings the system to the nearest metastable state and then noise or an applied field is needed to allow it to explore the energy landscape. Typical optimization methods are constructed by providing a suitable perturbation scheme on the states of the system. Recently hysteretic optimization was proposed [2] as an alternative to methods that use noise, such as simulated annealing [3]. Its basis is an analogy to a ferromagnetic demagnetization procedure: an external oscillating field with decreasing amplitude and low frequency is applied to the system, yielding at zero field the demagnetized state (DS), which is used as a reference state for material characterization [4].

The ferromagnetic random field Ising model (RFIM) has been extensively studied in literature as a paradigmatic example of disordered system [5], whose equilibrium and non-equilibrium properties are still tractable, though far from trivial. The RFIM is one of the simplest systems, where the crucial interplay between quenched disorder and exchange interaction gives rise in high enough dimensions to a disorder induced phase transition. This also affects the dynamics, where a non-equilibrium phase transition exists.

The equilibrium properties of the RFIM are governed by the $T = 0$ scaling [5] even at high temperatures. GS calculations have elucidated the properties of the phase diagram: In $d = 1$ the RFIM is trivially paramagnetic. In $d = 2$ there is no phase transition but an effective ferromagnetic regime for small systems. In $d \geq 3$ the GS displays an equilibrium phase transition induced by the disorder from a low disorder regime where the system is ferromagnetic (FM) to a strong disorder one, where the system is paramagnetic (PM) [1, 5]. The equilibrium critical exponents for random field magnets have been measured experimentally in $Fe_{0.93}Zn_{0.07}F_2$ [6, 7].

Likewise, the non–equilibrium properties of the RFIM have been studied by extensive numerical simulations and renormalization group calculations [8, 9, 10]. A disorder–induced transition is observed in the hysteretic behavior for $d \geq 3$: At low disorder the loop has a macroscopic jump in the magnetization, which disappears at a critical value of the disorder, above which the loop is smooth on a macroscopic scale. Numerical simulations [10] and renormalization group [9] have been used to estimate the critical exponents in various dimensions. A disorder induced non-equilibrium phase transition in the hysteresis loop has been studied experimentally in Co-CoO films [11] and Cu-Al-Mn alloys [12].

The relation between this non–equilibrium transition and the PM to FM one in the (equilibrium) GS has been debated in the past. Based on the similarity in the numerical values of the exponents and on mean–field equations, Maritan *et al.* [13] argued that the two transitions should be universal. Sethna *et al.* [14] refuted this due to the different natures of the two cases: the transition in the GS occurs for a zero external field, while the transition in the hysteresis loop occurs at the coercive field. More recently, the question of the universality of the exponents, with respect to the shape of the disorder distribution, was discussed in $d = 3$ simulations, mean-field theory, and on the Bethe lattice [15, 16, 17].

In this letter, we compare the equilibrium and non-equilibrium phase transitions in the RFIM, with evidence for universality. The key issue is the identification of a

reference non-equilibrium state, instead of focusing on the jump in the saturation loop. In the low disorder phase, a discontinuous hysteresis loop corresponds to a region of the field–magnetization plane not accessible by any field history [18, 19]. In this regime it is not possible to demagnetize completely by applying a slowly varying AC field. Thus one studies the DS as the state of lower (remanent) magnetization resulting from the demagnetization procedure. This state is uniquely defined, in the quasistatic limit, for any given realization of the random fields. It has two non-equilibrium phases: FM when the main loop has a jump, and PM otherwise. The remanent magnetization becomes the order parameter of the transition. Notice that the DS is defined at $H = 0$ and is therefore the natural non–equilibrium counterpart of the GS. This responds to the objection raised in Ref. [14] against the possible existence of universality.

We evaluate in $d = 3$ the finite–size scaling functions both for the equilibrium and non–equilibrium phase transition. By rescaling the disorder around the appropriate (distinct) critical values, the scaling functions can be collapsed onto the same curve using the same exponents values. However, the location of the critical point differs: the transition in the DS occurs at a lower disorder value. Thus there is an intermediate region where the GS is ferromagnetic but the DS is paramagnetic. To further corroborate our findings, we analyze the RFIM on the Bethe lattice: we compute the GS magnetization and compare it with the remanent magnetization of the DS [19]. While the exponents are the same in the two cases (coinciding with mean–field results), the Bethe lattice reproduces the ordering of the critical points in $d = 3$. Additional evidence for universality is obtained by comparing the order parameter distribution function at the critical point for finite systems.

In the RFIM, a spin $s_i = \pm 1$ is assigned to each site $i = 1, ...N$ (here of a cubic lattice in $d = 3$ or a Bethe one with coordination $z = 4$). The spins are coupled to their nearest–neighbors spins by a ferromagnetic interaction of strength J and to the external field H. In addition, to each site is associated a random field h_i taken from a Gaussian probability density $\rho(h) = \exp(-h^2/2R^2)/\sqrt{2\pi}R$, with width R, which measures the strength of the disorder. The Hamiltonian thus reads

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J s_i s_j - \sum_i (H + h_i) s_i \,, \qquad (1)$$

where $\sum_{\langle i,j \rangle}$ is restricted to nearest-neighbors pairs.

The RFIM GS is numerically solvable in a polynomial CPU–time with exact combinatorial algorithms. We find the GS via the min–cut/max–flow problem of combinatorial optimization, and use the so-called push–relabel variant of the preflow algorithm [20]. Such methods, properly implemented, are in general slightly non–linear in their performance as a function of the number of spins [1].

For the out of equilibrium case, we consider a simple relaxation dynamics obtained in the limit $T \to 0$ of the Glauber dynamics [8, 9, 10]: At each time step the spins align with the local effective field

$$s_i = \mathrm{sign}(J \sum_j s_j + h_i + H) \qquad (2)$$

until a metastable state is reached. To construct the hysteresis loop, the system is started from a state with all the spins down $s_i = -1$ and then H is ramped slowly from $H \to -\infty$ to $H \to \infty$. The limit of $dH/dt \to 0$ is taken after the limit $T \to 0$. In practice, this can be conveniently obtained by precise increases of the field, to always flip the first unstable spin. To reach the DS, the external field is changed through a nested succession $H = H_0 \to H_1 \to H_2 \to .....H_n... \to 0$, with $H_{2n} > -H_{2n+1} > H_{2n+2} > 0$, and $dH \equiv H_{2n} - H_{2n+2} \to 0$. This provides a perfect demagnetization with a uniquely defined DS ($dH/dt \to 0$). It being quite expensive computationally, we instead perform an approximate demagnetization using an algorithm discussed in Ref. [18] with $dH = 10^{-3}$. We verified that the states have negligible differences with perfectly demagnetized ones.

The RFIM critical exponents characterizing the disorder induced transition can be defined as usual: The magnetization $M \equiv \langle |m| \rangle$, with $m \equiv \sum_i s_i/N$, scales close to the transition point as $M = Ar^\beta$, where $r \equiv (R - R_c)/R_c$ is the reduced order parameter and A is a non-universal constant. The correlation length exponent $\xi = (Br)^{-\nu}$, where B is another non-universal constant, rules the finite–size scaling of the model

$$M = AL^{-\beta/\nu} f\left(BL^{1/\nu}(R - R_c)/R_c\right). \qquad (3)$$

GS simulations in $d = 3$ for Gaussian disorder, yield $1/\nu^{(GS)} \simeq 0.73$, $\beta^{(GS)} \simeq 0.02$ and $R_c^{(GS)} \simeq 2.28$ [21, 22, 23, 24].

The demagnetization process has been exactly solved in $d = 1$ [18] and on the Bethe lattice ([19], see also [25]). Numerical simulations in $d = 3$ indicate that the DS displays the same critical point as the saturation loop [26]. The transition point has been obtained numerically in $d = 3$ as $R_c^{(DS)} \simeq 2.16$ [10] and the critical exponents have been measured. E.g.Ref. [26] reports $\beta_{(DS)} = 0.04 \pm 0.02$ and $1/\nu_{(DS)} = 0.71 \pm 0.1$.

The numerical simulations for the GS and DS are done for the same disorder realizations for the both cases, for cubic lattices of linear sizes $L = 10, 20, 40, 80$. The results are averaged over several realizations of the quenched random fields. In both cases, we compute the average magnetization as a function of the disorder width. In Fig. 1 we collapse the two sets of data into a single curve, using two different values for R_c (i.e. $R_c^{(GS)} = 2.28$ and $R_c^{(DS)} = 2.16$) but the same values for the exponents (i.e. $1/\nu = 0.73$ and $\beta = 0.03$). The best value for the ratio of the non-universal constant is found to be $A_{DS}/A_{GS} \simeq 1$ and $B_{DS}/B_{GS} = 0.68 \pm 0.02$.

To provide another viewpoint and corroborate our claims, we compare the GS and DS on the Bethe lattice where analytical expressions can be found exactly.

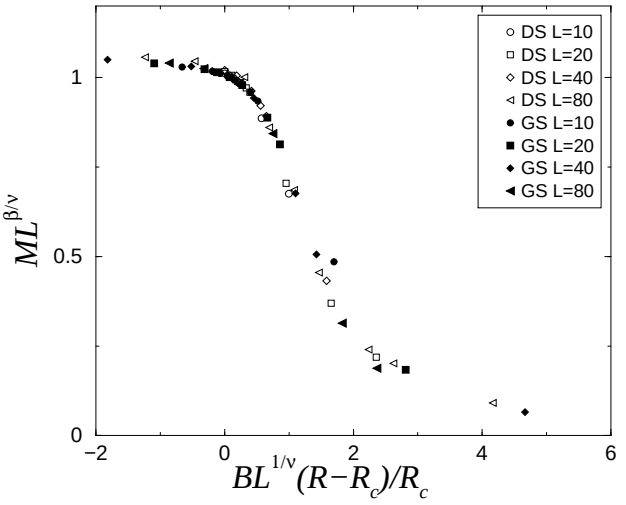

FIG. 1: Numerical results in $d =$: The magnetization can be collapsed using $R_c = 2.28$ (GS) and $R_c = 2.16$ (DS), $1/\nu = 0.73$ and $\beta = 0.03$. The scaling curve is the same for DS and GS indicating universal behavior. The values for the ratios of the non-universal constants are $A_{DS}/A_{GS} = 1$ and $B_{DS}/B_{GS} = 0.68$.

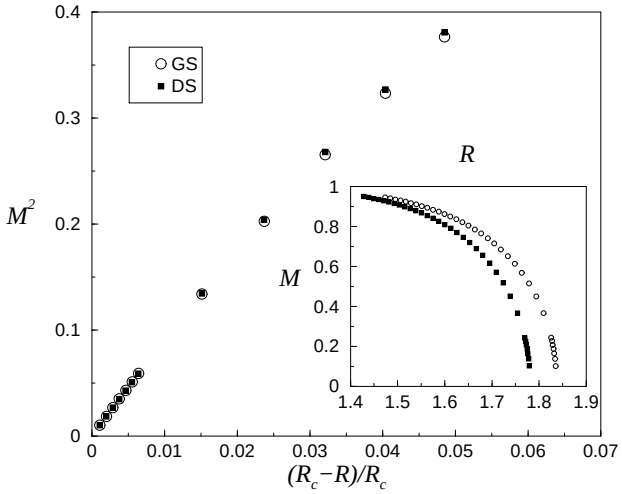

FIG. 2: The magnetization of the GS and the DS computed exactly on the Bethe lattice with $z = 4$ in the thermodynamic limit, showing the ordering of the critical point (see inset). When the data are plotted against the reduced parameter $(R_c - R)/R_c$ the curves superimpose. The result implies that for the Bethe lattice $A_{GS} = A_{DS}$.

The RFIM displays also on the Bethe lattice, for a large enough coordination number z, both an equilibrium and a non–equilibrium disorder induced phase transition [19]. To compare the GS and the DS around the respective transitions, we take directly the thermodynamic limit, using for the DS the results of Ref. [19]. We have obtained the GS magnetization following Refs. [27, 28] as

$$M = \int_{-\infty}^{\infty} dh\rho(h) \int_{-\infty}^{\infty} \prod_{k=1}^{z} dx_k W_\infty(x_k)\langle s_0\rangle. \qquad (4)$$

Here $W_\infty(x)$ is fixed-point probability distribution for the quantity $x_n \equiv \frac{T}{2}\ln(Z_n^+/Z_n^-)$, where $Z_n^\pm$ are defined as the partition functions of a branch of generation n with a fixed up (down) spin s_0 at the central site [27, 28], and is given by an implicit integral equation. The fixed point equation is solved by numerical integration, and the magnetization is computed for different values of R using Eq. (4), for $T = 0$ and $z = 4$. In Fig. 2 we show a comparison between the magnetization in the GS and in the DS ([19]).

As in $d = 3$ simulations, the transitions occur at two different locations (see the inset of Fig. 2), for $z = 4$ $R_c^{(DS)} = 1.781258...$ [19] and $R_c^{(GS)} \simeq 1.8375$, with the mean-field exponent ($\beta = 1/2$). When plotted against $(R - R_c)/R_c$ the two curves superimpose close to the critical point. This indicates that, though not required by universality, in the Bethe lattice $A_{GS} = A_{DS}$, as also found in $d = 3$. To investigate possible finite size scaling we have performed numerical simulations in the Bethe lattice, following the method of Ref. [29]. Collapsing the order parameter curve as in $d = 3$, using a scaling form similar to Eq. (3), does not appear to be possible in the Bethe lattice, because the scaling region is very narrow. Thus to test finite size scaling, we have computed the distribution of the magnetization m at the respective critical point, $R_c^{(DS)}$ and $R_c^{(GS)}$ for different lattice sizes N. The distributions can all be collapsed into the same curve (see Fig. 3), using the form $P(|m|) = f(|m|/M)/M$.

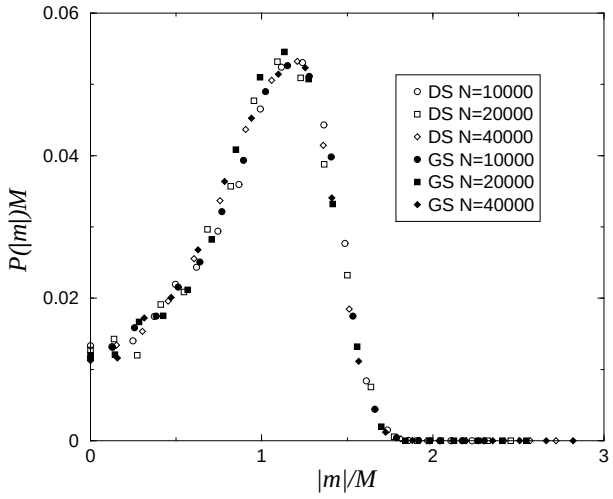

FIG. 3: The distributions of the magnetization in the DS and the GS at their respective critical points on the Bethe lattice, obtained numerically for different lattice sizes N, can be all collapsed together.

To conclude we provide evidence about the universality of the RFIM with Gaussian distributed disorder in and out of equilibrium. The key point is the identification of the correct order parameter for the non–equilibrium

transition. This quantity, the remanent magnetization of the DS, is the natural counterpart to the magnetization of the GS in the equilibrium case, in particular at a zero external field. Our results are based on a detailed numerical analysis in $d = 3$ and on an exact solution on the Bethe lattice. It would be interesting to confirm this conclusion by more complex measures, beside the order parameter, such as the scaling of the domain wall stiffness [24]. Regardless of the question of universality, the most intriguing point from our analysis is that the two transitions appear at different critical values of the disorder strength: the DS ferromagnetic phase is the first to disappear as the disorder is increased ($R_c^{(DS)} < R_c^{(GS)}$). The interpretation of the ordering is simply that the GS is as correlated as a state can be in the RFIM, due to the global optimization. Thus eg. as R is decreased it is natural that the FM correlations appear in the GS first.

Our results have important consequences on the use of the demagnetization as an optimization tool: the difference in the location of the critical points implies that for $R_c^{(DS)} \leq R \leq R_c^{(GS)}$, the DS (paramagnetic) is drastically different from the GS (ferromagnetic), suggesting that in that region hysteretic optimization is likely to fail. Moreover, to compare the GS and DS in a system, one expects to achieve the closest resemblance in this regime if the correlation length is the same; ie. due to the difference of the critical points at two separate values $R_1^{(DS)} < R_2^{(GS)}$, respectively – or as well two values of the effective coupling constant J in the Hamiltonian.

In addition to the ferromagnetic RFIM model, one can speculate about other systems where two disorder induced phase transitions exist. Numerical simulations and analytical results have shown that a disorder induced transition in the hysteresis loop can be observed in the random bond Ising model [30], in the random field O(N) model [31], in the random anisotropy model [32] and in the random Blume-Emery-Griffith model [30]. All these systems also show a transition in equilibrium and it would be interesting to compare their DS and GS. Another example would be the study of an interface in quenched disorder, where many results are known for the roughness exponent in and out of equilibrium (i.e. at the depinning threshold), and the results typically differ [5]. It would be interesting to measure the roughness of an interface after a demagnetization cycle (i.e. after the field driving the interface is cycled with decreasing amplitude), and compare its properties with those of the ground state interface. Finally, there is the issue of energetics of excitations in the respective ensembles: the universality of exponents and scaling functions would seem to imply that these also scale similarly.

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